Navigation Course in Mathematical Examples – Utilization of the Spherical Geometry

Navigacijski kurs u matematičkim primjerima – upotreba sferne geometrije

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Summary

The paper deals with applications of methods of spherical geometry in naval navigation. It summarises the essential knowledge of spherical geometry and presents some exercises and their solutions. These exercises can be applied in explanations of the basic principles of naval navigation.

KEY WORDS

navigation course
latitude and longitude
spherical geometry

Sažetak

U radu se govori o aplikacijama metoda sferne geometrije u brodskoj navigaciji. Dan je pregled osnovnih podataka o sfernoj geometriji uz nekoliko vježbi i njihova rješenja. Ove vježbe mogu se primijeniti pri pojašnjavanju osnovnih principa brodske navigacije.

KLJUČNE RIJEČI

navigacijski kurs
geografska širina i dužina
sferna geometrija

1. INTRODUCTION / *Uvod*

Euclid described the five basic axioms of geometry in his work Elements around the year 300 BC. Today, they are called the axioms of Euclidean geometry. You could say that Euclidean geometry describes the flat world. The description of the properties of Euclidean space gave rise to analytical geometry which facilitates approximations of the description of local properties of space and their applications using the Cartesian coordinate system. Such a description can be widely applied in solving various technical and physical issues, travel itineraries, etc.

Considerations over the validity of Euclid´s fifth axiom (the parallel postulate) changed significantly the view of the geometric perception of the world and influenced the modern development of geometry. Its disproof allowed the description of a world which is not flat (the so-called non-Euclidean geometry). The origin of non-Euclidean geometries falls into the first half of the 19th century and is connected with the names Bolyai and Lobachevsky. The non-Euclidean description of the world is dealt with by e.g. spherical and differential geometry. The development of knowledge of non-Euclidean geometry space introduced a number of applications. One of them is the description of curved space (Earth, the Universe …).

Spherical geometry is applied in the area of planning of long journeys on the Earth and mostly in naval and air navigation (c.f. [3], [4], [8], [9], [11] and [12]). Differential geometry studies curvature, transmission of properties by their display, geodetic curves and other properties of space (c.f. [2], [10]). For example, geodetic curves are curves which have zero geodetic curvature at each point. In other words, they are the curves along which movement is the "most economical" (time, distance or energy). Geodetic curves on a sphere are used in naval navigation as orthodromes (c.f. [9]). The methods of differential geometry allow to study the movements in long distances and are applied in air and naval navigation, in descriptions of the Universe (c.f. [2], [9], [10]). One of the best known methods of application of differential geometry is Einstein´s General Theory of Relativity.

The paper presents exercises (instructions and solutions) from naval navigation which applies the methods of spherical geometry. Let us note that the method of solution is presented in a manner comprehensible for secondary school graduates. The exercises may be used to explain the basic principles of naval navigation.

2. SPHERICAL GEOMETRY AND ITS APPLICATION TO NAVIGATION COURSE / *Sferna geometrija i njezina primjena na navigacijski kurs*

Earth's surface can be considered as spherical surface with the radius r = 6371*km*. The coordinates of points on the sphere are determined in the usual manner. Each point on the sphere (with the exception of the poles) has two geographic coordinates – the latitude and longitude. The latitude is denoted φ and longitude is denoted λ, evidently ; $\varphi \in \left\langle \frac{-\pi}{2}, \frac{\pi}{2} \right\rangle$; $\lambda \in (-\pi, \pi\rangle$, the coordinates of the point A (on the sphere) are written in the form $A = (\varphi, \lambda)$. The system of latitudes and longitudes generates an orthogonal coordinate network of parallels and meridians on the globe. Every main circle is the geodetic curve on the sphere. Finding the shortest route from the place to the place B is a basic task of spherical geometry. This route (i.e. the shortest route from A to B in the sentence above) is the length of the arc ($\leq \pi$) of the main circle which intersects points A, B.

To solve the examples some notions need to be defined. For more details see e.g. [8].

Definition 2.1 Let A, B be points on the sphere and k the main circle passing through points A, B. The size of the central angle ($\leq \pi$) relevant to arc AB of the main circle k is called the *spherical distance of points* A, B.

Definition 2.2 Let A, B, C be three different points on the sphere. The triangle ABC consisting of arcs $\widehat{AB}, \widehat{BC}, \widehat{AC}$ of main circles relevant to central angels less than or equal to π is called the *spherical triangle*. The spherical distances a, b, c of points B, C; A, C; A, B, respectively, are called the *sides* of the spherical triangle *ABC*. The angles between arcs $\widehat{AB}, \widehat{AC}$ in the point A, $\widehat{AB}$, $\widehat{BC}$ in the point B, $\widehat{BC}$, $\widehat{AC}$ in the point C are denoted α, β, γ, respectively and they are called the *inner angles of the spherical triangle*. The spherical triangle with at least one inner right angle is called the *right spherical triangle*.

Definition 2.3 Let's consider the point A on the sphere and the rotation, which turns the north point of horizon to the east. The *navigation course* is the angle φ of this rotation. The rotated point indicates the direction of movement in a given navigation course. The angle φ measures from 0° to 360°.

Definition 2.4 The number $|\pi - \varphi|$ is called *deviation of angle* φ (or *deviation of angle* φ *from right angle*).

The following theorems hold for spherical triangles:

Theorem 2.1 (*Spherical sinus theorem*) Let ABC be the spherical triangle, a, b, c be the spherical distances of points B, C; A, C; A, B and α, β, γ be the inner angles of spherical triangle ABC, respectively. Then the following forms are valid (1)

$$\begin{aligned} sina : sinb &= sin\alpha : sin\beta \\ sina : sinc &= sin\alpha : sin\gamma \\ sinb : sinc &= sin\beta : sin\gamma \end{aligned} \tag{1}$$

Remark 2.1 If the spherical triangle ABC is the right triangle (where $\gamma = \frac{\pi}{2}$) then equations (1) have the form

$$\begin{aligned} sina &= sin\alpha \cdot sinc \\ sinb &= sin\beta \cdot sinc \end{aligned} \tag{2}$$

Theorem 2.2 *(Spherical cosinus theorem)* Let ABC be the spherical triangle, a, b, c be the spherical distances of points B, C; A, C; A, B and α, β, γ be the inner angles of spherical triangle ABC, respectively. Then the following forms are valid

a) $\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos \alpha$
$\cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos \beta$
$\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos \gamma$

b) $\cos \alpha = -\cos \beta \cdot \cos \gamma + \sin \beta \cdot \sin \gamma \cdot \cos a$ (3)
$\cos \beta = -\cos \alpha \cdot \cos \gamma + \sin \alpha \cdot \sin \gamma \cdot \cos b$
$\cos \gamma = -\cos \alpha \cdot \cos \beta + \sin \alpha \cdot \sin \beta \cdot \cos c$

Remark 2.2 If the spherical triangle ABC is the right triangle (where $\gamma = \frac{\pi}{2}$) then equations (3) have form

a) $\cos c = \cos a \cdot \cos b$

b) $\cos \alpha = \sin \beta \cdot \cos a$ (4)
$\cos \beta = \sin \alpha \cdot \cos b$
$0 = -\cos \alpha \cdot \cos \beta + \sin \alpha \cdot \sin \beta \cdot \cos c$

Theorem 2.3 Let ABC be the spherical triangle, a, b, c be the spherical distances of the points B, C; A, C; A, B and α, β, γ be the inner angles of the spherical triangle ABC, respectively. Only one of the inner angles at most belongs to a quadrant other than the opposite side. It is the angle, which is opposite the side with the smallest deviation.

Proofs of these theorems are simple exercises from geometry; we can find them for example in [8].

Example 2.1 Let A, B be two places on the globe given by the latitude and longitude, $A = (\varphi_1, \lambda_1)$, $B = (\varphi_2, \lambda_2)$. Determine the shortest route d from A to B.

Solution: The spherical distance of these points has to be found and the arc length d is computed – the shortest route from A to B. They are three trivial cases – one of the points is the pole ($\varphi_1 = \pm\frac{\pi}{2}$ or $\varphi_2 = \pm\frac{\pi}{2}$), both points have the same longitude or the absolute value of the difference of longitudes is π.

In these cases, the points A, B lie on the same meridian or on the opposite meridians. Than the spherical distance of points A, B is equal to $|\varphi_1 - \varphi_2|$ and $d = r \cdot |\varphi_1 - \varphi_2|$.

Now, suppose points A, B are not in the above-mentioned special positions. Let's consider the spherical triangle ABC, where C is the North Pole. Denote a the spherical distance of points B, C, b the spherical distance of points A, C and c the spherical distance of points A, B. Clearly $a = \frac{\pi}{2} - \varphi_2$, $b = \frac{\pi}{2} - \varphi_1$ (and with respect to Definition 2.2 $0 < a < \pi$, $0 < b < \pi$). The angle $\gamma = |\lambda_1 - \lambda_2|$ if $0 < |\lambda_1 - \lambda_2| < \pi$ or $\gamma = 2\pi - |\lambda_1 - \lambda_2|$ if $\pi < |\lambda_1 - \lambda_2| < 2\pi$ (Fig. 1). It means that $0 < \gamma < \pi$ and $cos\ \gamma = cos|\lambda_1 - \lambda_2| = cos(\lambda_1 - \lambda_2)$.

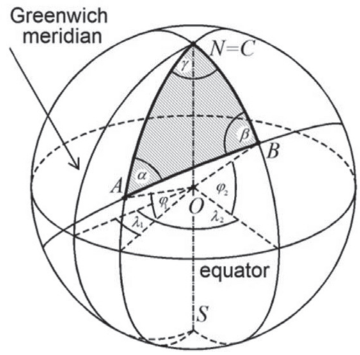

Figure 1 The spherical distance c of points A, B is computed (Theorem 2.2).
Slika 1. Izračunata je sferna udaljenost c od točaka A, B (Teorem 2.2)

$$\begin{aligned} cosc &= cosa \cdot cosb + sina \cdot sinb \cdot cos\gamma \\ cosc &= sin\varphi_1 \cdot sin\varphi_2 + cos\varphi_2 \cdot cos(\lambda_1 - \lambda_2) \\ c &= cos^{-1}(cosc) \end{aligned} \tag{5}$$

The shortest route from the point A to the point B is $d = r \cdot c$, where r is the radius of Earth.

Now the angles α, β are computed (Theorem 2.1, Theorem 2.3)

$$\begin{aligned} \sin \alpha &= \frac{\sin a}{\sin c} \cdot \sin \gamma, \\ \sin \beta &= \frac{\sin b}{\sin c} \cdot \sin \gamma. \end{aligned} \tag{6}$$

We obtain two angles for α, β, but only one of them is allowable (Theorem 2.3). The angle α (β) determines the navigation course to move from the point A (B) to the point B (A) along the main circle.

Example 2.2 The ship sails from Rio de Janeiro (22°55′S, 43°09′W) to Lisbon via the shortest route.

How long does the voyage take if the ship sails at an average speed $15NMh^{-1}$ (nautical miles[1] per hour). What are the navigation courses of departure and arrival?

At which point the ship crosses the equator?

Solution: We denote $A = (\varphi_1, \lambda_1) = (-22°55′, -43°09′)$ as the point of departure, $B(\varphi_2, \lambda_2) = (38°42′, -9°11′)$ as the point of arrival, C as the North Pole and $M = (\varphi_3, \lambda_3)$ as the point where the ship crosses the equator (Fig. 2).

[1] A nautical mile (NM) is a unit of measurement defined as exactly. It is defined as one minute of latitude, which is equivalent to one sixtieth of a degree of latitude (Source: https://en.wikipedia.org/wiki/Units_of_measurement)

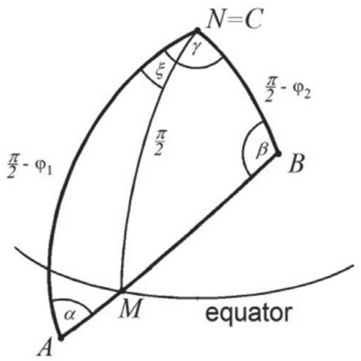

Figure 2

Slika 2

a) The spherical distance of points A, C is $b = \frac{\pi}{2} - \varphi_1 \doteq 1{,}9708rad$ ($\varphi_1 = -22°55' \doteq -0{,}4rad$), the spherical distance of points B, C is $a = \frac{\pi}{2} - \varphi_2 \doteq 0{,}8954rad$ ($\varphi_2 = 38°42' \doteq 0{,}6754rad$). (Example 2.1). The spherical distance c of points A, B is computed from equations (3), where $\lambda_1 = -43°09' \doteq -0{,}7531rad$, $\lambda_2 = -9°11' \doteq -0{,}1603rad$ and $\gamma = \lambda_2 - \lambda_1 = 0{,}5928\ rad$. We obtain

$$cosc = cos(\tfrac{\pi}{2} - \varphi_2) \cdot cos(\tfrac{\pi}{2} - \varphi_1) + sin(\tfrac{\pi}{2} - \varphi_2) \cdot sin(\tfrac{\pi}{2} - \varphi_1) \cdot cos\gamma$$

$$cosc = sin\varphi_2 \cdot sin\varphi_1 + cos\varphi_2 \cdot cos\varphi_1 \cdot cos\gamma$$

$$cosc \doteq 0{,}3527$$

$$c \doteq 1{,}2103rad$$

The distance d of points A, B is $d = r \cdot c$, i. e. d $\doteq 7711km \doteq 4163{,}62NM$.

The voyage time t is computed from the known form $t = \frac{d}{v}$, where v is average speed. Therefore $t = \frac{4163{,}62}{15} \doteq 227{,}5747h \doteq 11days14h$.

The angles α, β are computed from equations (6). The side c of the spherical triangle ABC has the smallest deviation from the right angle, therefore only the angle γ belongs to a quadrant other than it's opposite side. We obtain $sin\alpha \doteq 0{,}4463$ and $\alpha \doteq 0{,}466rad \doteq 27°46'26''$, $sin\beta \doteq 0{,}55$ and $\beta \doteq 0{,}5823rad \doteq 33°21'51''$. Navigation course on arrival is $\pi - \beta \doteq 146°38'09''$.

Result: The navigation course at the departure was 27°46´26", the navigation course at the arrival was 146°38´09". The ship travelled $7711km$ ($4163{,}62NM$) in $11days14h$.

b) Now let's consider the spherical triangle AMC, spherical distance of points M, C is $b' = \frac{\pi}{2}$. ξ is denoted as the inner angle at vertex C and x as the side opposite to angle ξ (Fig. 2). The equations (4) are applied on the triangle AMC, we obtain successively

$$cos\tfrac{\pi}{2} = cosx \cdot cos(\tfrac{\pi}{2} - \varphi_1) + sinx \cdot sin(\tfrac{\pi}{2} - \varphi_1) \cdot cos\alpha$$

$$x \cdot sin\,\varphi_1 + sinx \cdot cos\,\varphi_1 \cdot cos\alpha$$

$$0 = cos$$

Since $\varphi_1 \neq \frac{\pi}{2}$, $\alpha \neq \frac{\pi}{2}$, $x \neq \frac{\pi}{2}$ and $x < c < \frac{\pi}{2}$ also we obtain

$$tanx = \frac{-tan\varphi_1}{cos\alpha} \doteq 0{,}4778$$

The length d' of the way route A to M is $d' = r \cdot x \doteq 2840km \doteq 1533{,}35NM$.

The angle ξ at the vertex C is computed from forms (1), Theorem 2.1.

$$\frac{sin\xi}{sin\alpha} = \frac{sinx}{sin\frac{\pi}{2}}$$

$$sin\,\xi = sinx \cdot sin\alpha \doteq 0{,}4268 \cdot 0{,}4447 \doteq 0{,}2009$$

$$\xi \doteq 0{,}2022rad \doteq 11°27'28''.$$

It means that the longitude of the point M is $\lambda_3 = \lambda_1 + \xi = -0{,}5508 \doteq -31°33'31''$, i. e. $31°33'31''$ of the west longitude.

Result: The ship crosses the equator at the point M whose – longitude is $31°33'31''W$ and its distance from the point A is $2840km$ ($1533{,}35NM$).

Example 2.3 The ship still sails on the main circle and it crosses the equator at the angle $\alpha = 50°$ and the parallel l of latitude $\varphi = 15°$ at an angle ψ. Compute the angle $\psi \left(\leq \frac{\pi}{2}\right)$.

Solution: A is denoted as the point in which the ship crosses the equator e, B as the point in which the ship crosses the parallel l, m as the meridian of the point B. We can consider the right spherical triangle ABC, where $C = e \cap m$ (Fig.3). Clearly $\alpha = \varphi$ and $\varphi = 15° \doteq 0{,}2618\ rad$, $\alpha = 50° \doteq 0{,}8727rad$ are given.

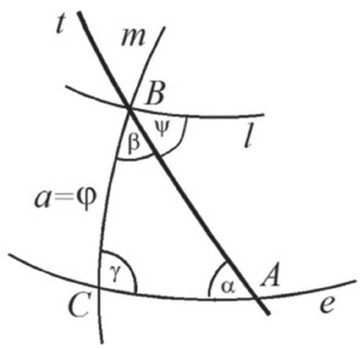

Figure 3

Slika 3.

We obtain from equations (4) $sin\beta = \frac{cos\alpha}{cos\alpha} \doteq 0{,}6655$, we obtain two angles satisfying the equation, $\beta_1 \doteq 0{,}7281\ rad \doteq 41°43'04''$, $\beta_2 \doteq 2{,}4135 \doteq 138°16'56''$. Since the parallel and meridians intersect orthogonally, it is obvious that $\psi = \frac{\pi}{2} - \beta_1 = \beta_2 - \frac{\pi}{2} = 48°16'56''$.

Example 2.4 Two ships detected the telegraph signal SOS. The first ship is at the point $A = (24°45'\ N, 138°18'E)$ and it has the navigation course $\omega_1 = 72°38'$, the second ship is at the point $B = (15°27'N, 175°34'E)$ and it has the navigation course $\omega_2 = 360° - 8°\ 53'$. Compute latitude φ and longitude λ of the point C, where the ship in danger is located (Fig. 4).

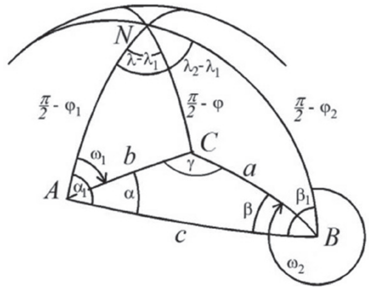

Figure 4

Slika 4

Solution: Three spherical triangles ABN, ABC and ACN are considered, N is the North Pole. The two sides of the spherical triangle ABN are $\frac{\pi}{2} - \varphi_1$, $\frac{\pi}{2} - \varphi_2$ and the inner angle at the point N is $\lambda_2 - \lambda_1$ (Fig. 4). Its side c and inner angles α_1, β_1 are computed (Theorem 2.1). $\varphi_1 = 24°45' \doteq 0{,}423rad$, $\varphi_2 = 15°27' \doteq 0{,}2697rad$, $\lambda_1 = 138°18' \doteq 2{,}4138rad$, $\lambda_2 = 175°34' \doteq 3{,}0642rad$, $\omega_1 = 72°38' \doteq 1{,}2677rad$, $\omega_2 = 351°7' \doteq 6{,}1281rad$ are known.

$$cosc = cos(\tfrac{\pi}{2} - \varphi_1) \cdot cos(\tfrac{\pi}{2} - \varphi_2) + sin(\tfrac{\pi}{2} - \varphi_1) \cdot sin(\tfrac{\pi}{2} - \varphi_2) \cdot cos(\lambda_2 - \lambda_1)$$

$$cosc \doteq 0{,}8081$$

$$c \doteq 0{,}6298rad$$

Next we have

$$cos(\tfrac{\pi}{2} - \varphi_2) = cos(\tfrac{\pi}{2} - \varphi_1) \cdot cosc + sin(\tfrac{\pi}{2} - \varphi_1) \cdot sinc \cdot cos\alpha_1$$

$$cos\alpha_1 = \frac{sin\varphi_2 - sin\varphi_1 \cdot cosc}{cos\varphi_1 \cdot sinc}$$

$$cos\alpha_1 \doteq -0{,}1345$$

$$\alpha_1 \doteq 1{,}7057rad$$

$$cos(\tfrac{\pi}{2} - \varphi_1) = cos(\tfrac{\pi}{2} - \varphi_2) \cdot cosc + sin(\tfrac{\pi}{2} - \varphi_2) \cdot sinc \cdot cos\,\beta_1$$

$$cos\beta_1 = \frac{sin\varphi_1 - sin\varphi_2 \cdot cosc}{cos\varphi_2 \cdot sinc}$$

$$cos\beta_1 \doteq 0{,}2095$$

$$\beta_1 \doteq 1{,}7819rad$$

The spherical triangle has sides a, b, c, inner angles α, β, γ. The side c has been computed and the inner angels α, β can be determined from the given navigation courses ω_1, ω_2, clearly $\alpha = \alpha_1 - \omega_1 \doteq 0{,}4380rad$, $\beta = \beta_1 - (2\pi - \omega_2) = \beta_1 + \omega_2 - 2\pi \doteq 1{,}6268rad$.

Now the inner angle γ at the vertex C (Theorem 2.2 b)) and sides a, b are computed (Theorem 2.1)

$$cos\gamma = -cos\alpha \cdot cos\beta + sin\alpha \cdot sin\beta \cdot cosc$$

$$cos\gamma \doteq -0{,}083$$

$$\gamma \doteq 0{,}3923rad$$

$$sina = \frac{sin\alpha}{sin\gamma} \cdot sinc \doteq 0{,}2717$$

$$\alpha \doteq 0{,}2751rad$$

$$sinb = \frac{sin\beta}{sin\gamma} \cdot sinc \doteq 0{,}6395$$

$$b \doteq 0{,}6939rad$$

Finally the spherical triangle ACN is considered and its side $\frac{\pi}{2} - \varphi$ is computed (Theorem 2.2) (φ is the latitude of the point C) and its inner angle $\lambda - \lambda_1$, where λ is longitude of the point C (Theorem 2.1).

$$cos(\tfrac{\pi}{2} - \varphi)=cosb \cdot cos(\tfrac{\pi}{2} - \varphi_1) + sinb \cdot sin(\tfrac{\pi}{2} - \varphi_1) \cdot cos\omega_1$$
$$sin\varphi = cosb \cdot sin\varphi_1 + sinb \cdot cos\varphi_1 \cdot cos\omega_1$$
$$sin\varphi \doteq 0{,}4952$$
$$\varphi \doteq 0{,}5181 rad \doteq 29°41'01''$$

$$sin(\lambda - \lambda_1) = \frac{sinb \cdot cos\varphi}{sin\omega_1}$$
$$sin(\lambda - \lambda_1) \doteq 0{,}5821$$
$$\lambda - \lambda_1 \doteq 0{,}6213 rad$$
$$\lambda \doteq 3{,}0351 rad \doteq 173°54'$$

Result: The ship in danger has latitude $29°41'01''N$ and and longitude $173°54'E$.

Remark 2.3 The shortest route from the point A to the point B on the sphere (also so called great-circle distance) is the main circle. But the ship sailing along this circle still changes the navigation course, which is unfavourable for shipping as each great-circle distance is approximated (piecewise) by some (arcs of) rhumb lines. A rhumb line is a curve that intersects all meridians at equal angles. It means a ship sailing along a rhumb line has a fixed navigation course. When the great-circle distance is approximated by some rhumb lines then the ship has a fixed navigation course for each of the rhumb lines.

3. CONCLUSION / *Zaključak*

The paper presents exercises in naval navigation and their solutions. They were deliberately selected in a way which is comprehensible for people familiar with secondary school mathematics. They can be used to describe the principles of naval navigation and to make lessons of mathematics more interesting at both secondary and tertiary schools. Students need to learn gradually the possible applications of the mathematical apparatus. Later, they might encounter various applications based in the mathematical apparatus (albeit not apparent at first sight). Among the interesting applications in the areas of transport and navigation belong [1], [5] - [7], [13].

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